

On how to complete the Dynamical Generation of Quark-Level Linear Sigma Model like Theories beyond one Loop

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Abstract. A self-consistent strategy is proposed to complete in a renormalization scheme independent way the dynamical generation of Quark-Level Linear Sigma Model like Lagrangean theories beyond one loop like the theories of strong and electroweak interactions. The present discussion refers for simplicity to scalar and pseudoscalar degrees of freedom only while disregarding yet — without loss of generality — vector and axial vector degrees of freedom. Moreover points the discussion to approximations underlying dimensional and implicit regularization as presently used.

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DYNAMICAL GENERATION OF LAGRANGEAN THEORIES

Throughout the construction of Lagrangean densities used e.g. in particle physics one faces at least two problems: 1) unpredictable Lagrangeans contain too many uncorrelated parameters (masses, couplings) which have to be fitted to experiment; 2) inherent divergencies of logarithmic, linear, quadratic, ... type need to be renormalized. The concept of *dynamical generation* [1] of Lagrangean theories addresses and solves both issues simultaneously:

- 1) In the spirit of Eguchi [2] one starts out from very few fundamental 3-point interaction vertices and constructs then on the basis of these vertices by “loop-shrinking” [1] the so-called effective action (and its underlying Lagrangean) containing also terms for all remaining n -point vertices between the fields making up the theory.
- 2) The couplings of the fundamental 3-point interaction vertices are then chosen such that linear, quadratic [4, 5], ... divergencies cancel [1] while the remaining logarithmic divergencies are renormalized [8] by adding to the effective action counter terms which replace in the spirit of the log.-divergent gap equation of Delbourgo and Scadron (DS) [6, 7] the integral $\int d^4p (p^2 - \bar{m}^2)^{-2}$ at some experimentally defined renormalization scale \bar{m} by some universal complex number.

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RENORMALIZATION OF LOGARITHMIC DIVERGENCIES

Throughout the manuscript we shall apply the renormalization procedure of DS [6, 7] and replace the log.-divergent Bosonic one-loop integral at some renormalization scale \bar{m} (being in the case of DS approximately equal to the nonstrange constituent quark mass \hat{m} , i.e. $\bar{m} \simeq \hat{m} = m_q$) by the finite number $+\frac{i}{16\pi^2}$ by adding suitable counterterms to the effective action. Hence we shall perform in all log.-divergent integrals the replacement

$$I_2(\bar{m}^2) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - \bar{m}^2)^2} \rightarrow +\frac{i}{16\pi^2}, \quad (1)$$

being known as the *log.-divergent gap equation* [6, 7, 9]. The replacement can be understood as the analytical continuation of the following integral identity (see also Eq. (29) in Appendix A) to the the exponent $n = 2$:

$$I_n(\bar{m}^2) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - \bar{m}^2)^n} \stackrel{n \geq 3}{=} (-1)^n \frac{i}{16\pi^2} \frac{1}{(n-1)! \bar{m}^{2n-4}}. \quad (2)$$

To apply the renormalization merely in integrals with zero external four-momentum (“local integrals”) we recall here that it has been pointed out in the context of the so-called implicit regularization scheme [10, 11] that divergent integrals with nonvanishing external four-momentum (“non-local integrals”) can be decomposed in equally divergent local integrals and less divergent non-local integrals by repeated application of the identity $1/((p-k)^2 - m^2) = 1/(p^2 - m^2) - (k^2 - 2k \cdot p)/[(p^2 - m^2)((p-k)^2 - m^2)]$. After making divergent integrals “local” a subsequent repeated application of the identity

$$\frac{1}{p^2 - m^2} = \frac{1}{p^2 - \bar{m}^2} + \frac{m^2 - \bar{m}^2}{(p^2 - \bar{m}^2)(p^2 - m^2)} \quad (3)$$

to the local limit of propagators in the integrands will then isolate all quadratic and logarithmic divergencies such that they can either be renormalized by applying the log.-divergent gap equation or by cancelling quadratic divergencies at the renormalization scale \bar{m} . Following this prescription we obtain with the help of the integral identities listed in Appendix A e.g.:

$$\begin{aligned} I_1(m^2) &= I_1(\bar{m}^2) + (m^2 - \bar{m}^2) \underbrace{I_2(\bar{m}^2)}_{\rightarrow \frac{i}{16\pi^2}} + (m^2 - \bar{m}^2)^2 I_{2,1}(\bar{m}^2, m^2) \\ &\rightarrow I_1(\bar{m}^2) + \frac{i}{16\pi^2} (m^2 - \bar{m}^2) \left(2 - \frac{m^2}{m^2 - \bar{m}^2} \ln \frac{m^2}{\bar{m}^2} \right), \end{aligned} \quad (4)$$

$$\begin{aligned} I_2(m^2) &= \underbrace{I_2(\bar{m}^2)}_{\rightarrow \frac{i}{16\pi^2}} + 2(m^2 - \bar{m}^2) I_{2,1}(\bar{m}^2, m^2) + (m^2 - \bar{m}^2)^2 I_{2,2}(\bar{m}^2, m^2) \\ &\rightarrow \frac{i}{16\pi^2} \left(1 - \ln \frac{m^2}{\bar{m}^2} \right), \end{aligned} \quad (5)$$

$$I_{1,1}(\bar{m}^2, m^2) = \underbrace{I_2(\bar{m}^2)}_{\rightarrow \frac{i}{16\pi^2}} + (m^2 - \bar{m}^2) I_{2,1}(\bar{m}^2, m^2) \rightarrow \frac{i}{16\pi^2} \left(2 - \frac{m^2}{m^2 - \bar{m}^2} \ln \frac{m^2}{\bar{m}^2} \right), \quad (6)$$

$$\begin{aligned} I_{1,1}(m_1^2, m_2^2) &= \frac{1}{2} \left(I_{1,1}(\bar{m}^2, m_1^2) + I_{1,1}(\bar{m}^2, m_2^2) + (m_1^2 + m_2^2 - \bar{m}^2) I_{1,1,1}(m_1^2, m_2^2, \bar{m}^2) \right) \\ &\rightarrow \frac{i}{32\pi^2} \left(4 - \frac{m_1^2}{m_1^2 - \bar{m}^2} \ln \frac{m_1^2}{\bar{m}^2} - \frac{m_2^2}{m_2^2 - \bar{m}^2} \ln \frac{m_2^2}{\bar{m}^2} \right. \\ &\quad \left. + (m_1^2 + m_2^2 - \bar{m}^2) \frac{m_1^2 m_2^2 \ln \frac{m_1^2}{m_2^2} + m_2^2 \bar{m}^2 \ln \frac{m_2^2}{\bar{m}^2} + \bar{m}^2 m_1^2 \ln \frac{\bar{m}^2}{m_1^2}}{(m_1^2 - m_2^2)(m_2^2 - \bar{m}^2)(\bar{m}^2 - m_1^2)} \right). \quad (7) \end{aligned}$$

Analogously, the renormalization of the equal mass sunset diagram with zero external momentum is performed in Appendix B.

THE QUARK-LEVEL LINEAR SIGMA MODEL (QLL σ M)

For various reasons like e.g. the still lacking [12, 14, 13] evidence for the existence of gluons and new developments in mathematical physics there has developed an alternative approach to strong interactions being different from Quantum Chromodynamics (QCD) which is known as the so-called QLL σ M.² The spin 1/2 Fermions of the QLL σ M, i.e. the (anti)quarks, are not interacting via gluons like in QCD, yet via mesons described by Bosonic scalar, pseudoscalar, vector and axial-vector fields. In order to outline the idea of how to perform a complete dynamical generation of QLL σ M like Lagrangeans beyond one loop order it is not necessary to consider the full-fledged Lagrangean of the $U(6) \times U(6)$ QLL σ M [1, 13, 18], yet one can either restrict one-self to the much simpler Lagrangean of the $SU(2) \times SU(2)$ QLL σ M studied thoroughly to one loop e.g. by DS [6, 7] or the $U(1) \times U(1)$ QLL σ M being intimately related to the (super-symmetric) Wess-Zumino model [20]. The apparent similarity of the field content of the Lagrangeans of the QLL σ M and electroweak interactions allows to transfer and extend the ideas of this manuscript explained within the context of the QLL σ M in a straight forward way to the theory of the electroweak force and to combine the former and the latter to a new standard model of particle physics.

Disregarding here for simplicity vector and axial vector mesons the $SU(2) \times SU(2)$ QLL σ M assuming $N_F = 2N_c = 6$ Fermions, one scalar isoscalar meson σ and $N_\pi =$

² Historically it has been probably Lévy [15] the first to add Fermions, i.e. nucleons, to the Linear Sigma Model (L σ M) of Schwinger, Gell-Mann and Lévy [16] while Cabbibo and Maiani [17] presumably were the first to replace the nucleons by quarks. The name QLL σ M has been coined by Delbourgo and Scadron [6, 7] who undertook on the basis of dimensional regularization a dynamical generation of the QLL σ M just to one loop. It has been then the author of the manuscript to point out that the experimentally favoured asymptotically free phase of QLL σ M belongs to the well-acceptable class of theories being non-Hermitian, yet PT-symmetric [1, 13, 18][19].

3 pions is constructed on the basis the interaction Lagrangean $\mathcal{L}_{\text{quark-meson}}(x) = g \bar{q}_+^c(x) (\sigma(x) + i \gamma_5 \vec{\tau} \cdot \vec{\pi}(x)) q_-(x)$ [6, 7, 22] yielding by loop-shrinking the following leading terms in the Lagrangean of the effective action for meson-meson interactions :

$$\mathcal{L}_{\text{meson-meson}} = g_{\sigma\pi\pi} \sigma(x) (\sigma(x)^2 + \vec{\pi}(x)^2) - \frac{\lambda}{4} (\sigma(x)^2 + \vec{\pi}(x)^2)^2 + \dots \quad (8)$$

Analogously the $U(1) \times U(1)$ QLL σ M assuming N_F Fermions, one scalar isoscalar meson σ and one isoscalar pseudoscalar meson η is constructed on the basis the interaction Lagrangean $\mathcal{L}_{\text{quark-meson}}(x) = g \bar{q}_+^c(x) (\sigma(x) + i \gamma_5 \eta(x)) q_-(x)$ yielding by loop-shrinking the following leading terms in the Lagrangean of the effective action for meson-meson interactions:

$$\mathcal{L}_{\text{meson-meson}} = g_{\sigma\eta\eta} \sigma(x) (\sigma(x)^2 + \eta(x)^2) - \frac{\lambda}{4} (\sigma(x)^2 + \eta(x)^2)^2 + \dots \quad (9)$$

Although we are going to present in what follows analytical results for the $SU(2) \times SU(2)$ QLL σ M only, the analogous results for the $U(1) \times U(1)$ QLL σ M are easily recovered by setting $N_\pi = 1$ and performing the replacements $g_{\sigma\pi\pi} \rightarrow g_{\sigma\eta\eta}$ and $\vec{\pi}^2 \rightarrow \eta^2$.

DYNAMICAL GENERATION OF THE $SU(2) \times SU(2)$ QLL σ M

Following the formalism described in Ref. [1] the relevant terms in the effective action of the $SU(2) \times SU(2)$ QLL σ M for the σ -one-point function (see Fig. 1), for the two-point function of the quarks (see Fig. 2), of the σ (see Fig. 3) and of the pions (see Fig. 4) are obtained in Appendix C. Instead of determining directly the effective meson-meson-interaction couplings $g_{\sigma\pi\pi}$ and λ completely by loop shrinking as a function of the quark-meson coupling g , we shall take here a different strategy and try to obtain the functional relation between the three couplings by direct elimination of quadratic divergencies in the effective action. Recalling the quadratic divergence of the sunset/sunrise diagram as discussed in Appendix B we extract the quadratically divergent part of the effective actions of Appendix C with the following result:

$$S_{(1)}[\sigma] = \int d^4x \sigma(x) \times \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} i \left\{ -4g N_F m_q + \left(1 - \frac{\lambda}{4\pi^2}\right) g_{\sigma\pi\pi} (3 + N_\pi) \right\} + \dots \quad (10)$$

$$S_{(2)}[\bar{q}q] = \frac{i}{2} \int d^4x \bar{q}_+^c(x) q_-(x) \times \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} \frac{2g}{m_\sigma^2} \left\{ -4g N_F m_q + \left(1 - \frac{\lambda}{4\pi^2}\right) g_{\sigma\pi\pi} (3 + N_\pi) \right\} + \dots \quad (11)$$

$$S_{(3)}[\sigma^2] = \frac{i}{2} \int d^4x \sigma(x)^2 \times \left[\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} \frac{6g_{\sigma\pi\pi}}{m_\sigma^2} \left\{ -4g N_F m_q + \left(1 - \frac{\lambda}{4\pi^2}\right) g_{\sigma\pi\pi} (3 + N_\pi) \right\} \right]$$

$$+ \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} \left\{ 4g^2 N_F - \left(1 - \frac{\lambda}{4\pi^2}\right) \lambda (3 + N_\pi) \right\} \Big] + \dots \quad (12)$$

$$\begin{aligned} S_{(4)}[\vec{\pi}^2] &= \frac{i}{2} \int d^4 x \, \vec{\pi}(x)^2 \\ &\times \left[\int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} \frac{2g_{\sigma\pi\pi}}{m_\sigma^2} \left\{ -4g N_F m_q + \left(1 - \frac{\lambda}{4\pi^2}\right) g_{\sigma\pi\pi} (3 + N_\pi) \right\} \right. \\ &\left. + \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} \left\{ 4g^2 N_F - \left(1 - \frac{\lambda}{4\pi^2}\right) \lambda (3 + N_\pi) \right\} \right] + \dots \quad (13) \end{aligned}$$

There are now (at least) three options to proceed:

Option 1: *dynamical generation by complete elimination of quadratic divergencies*

Elimination of quadratic divergencies in the tadpole-sum results in Eq. (14), while the elimination of quadratic divergences in the remaining part of the σ - and π -self-energy yields Eq. (15):

$$0 = -4g N_F m_q + \left(1 - \frac{\lambda}{4\pi^2}\right) g_{\sigma\pi\pi} (3 + N_\pi) \quad (14)$$

$$0 = +4g^2 N_F - \left(1 - \frac{\lambda}{4\pi^2}\right) \lambda (3 + N_\pi) \quad (15)$$

The system of equations can be solved for λ and $g_{\sigma\pi\pi}$ as a function of g and m_q :

$$0 = \frac{\lambda^2}{4\pi^2} - \lambda + \frac{4g^2 N_F}{3 + N_\pi} \Rightarrow \lambda = 2\pi^2 \left(1 \pm \sqrt{1 - \frac{4g^2 N_F}{\pi^2(3 + N_\pi)}} \right) \quad (16)$$

$$\frac{4g N_F m_q}{g_{\sigma\pi\pi}} = \frac{4g^2 N_F}{\lambda} \Rightarrow g_{\sigma\pi\pi} = \lambda \frac{m_q}{g} \quad (17)$$

$$\Rightarrow g_{\sigma\pi\pi} = 2\pi^2 \left(1 \pm \sqrt{1 - \frac{4g^2 N_F}{\pi^2(3 + N_\pi)}} \right) \frac{m_q}{g} \quad (18)$$

It's interesting to note that the identity $g_{\sigma\pi\pi} = -e^{i(\alpha-\beta)} (m_\sigma^2 - m_\pi^2)/(2|f_\pi|)$ of the $\text{L}\sigma\text{M}$ in combination with Eq. (17) would imply the following generalized NJL-relation:

$$m_\sigma^2 = m_\pi^2 + (2m_q)^2 \left(-\frac{|g|f_\pi}{m_q} \right) \left(\frac{\lambda}{2g^2} \right) = m_\pi^2 + |2m_q|^2 \left(-\frac{|g|f_\pi}{m_q} \right) \left(\frac{\lambda}{2|g|^2} \right) e^{2i(\beta-\alpha)}, \quad (19)$$

which turns — presuming the Golberger-Treiman (GT) relation on the quark-level $m_q \approx -|g|f_\pi$ and some eventually complex-valued quark-meson-coupling constant $g = |g|e^{i\alpha}$ and some complex-valued decay constant $f_\pi = |f_\pi|e^{i\beta}$ — into the standard NJL-relation $m_\sigma^2 \approx m_\pi^2 + |2m_q|^2$ for $\lambda \approx 2|g|^2 e^{2i(\alpha-\beta)}$ to be confronted with Eq. (16).

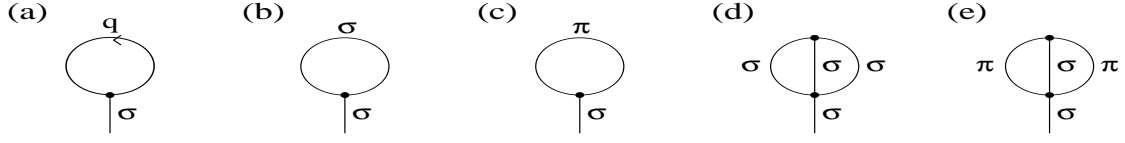


FIGURE 1. Tadpole sum: contributions to the σ one-point function

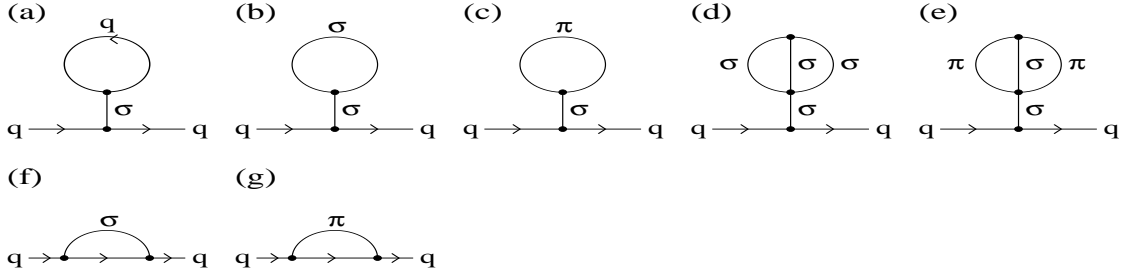


FIGURE 2. Quark mass: contributions to the quark self-energy

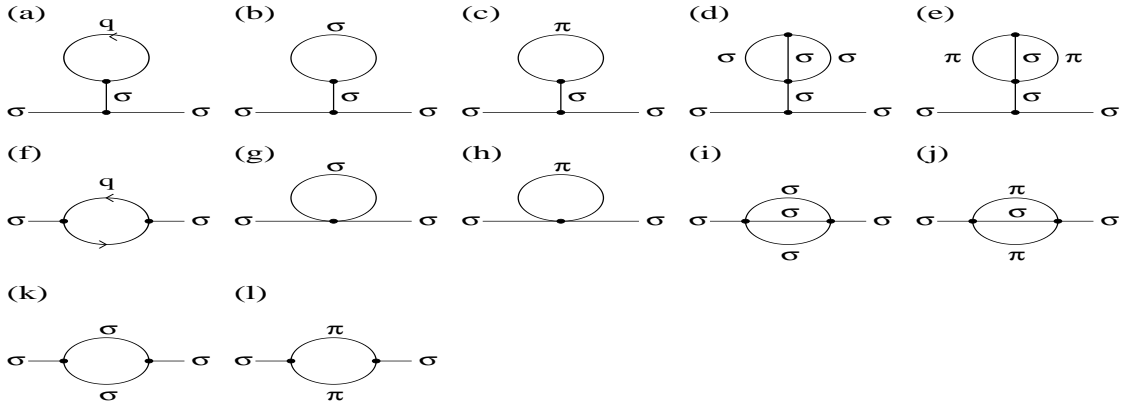


FIGURE 3. Sigma mass: contributions to the σ self-energy

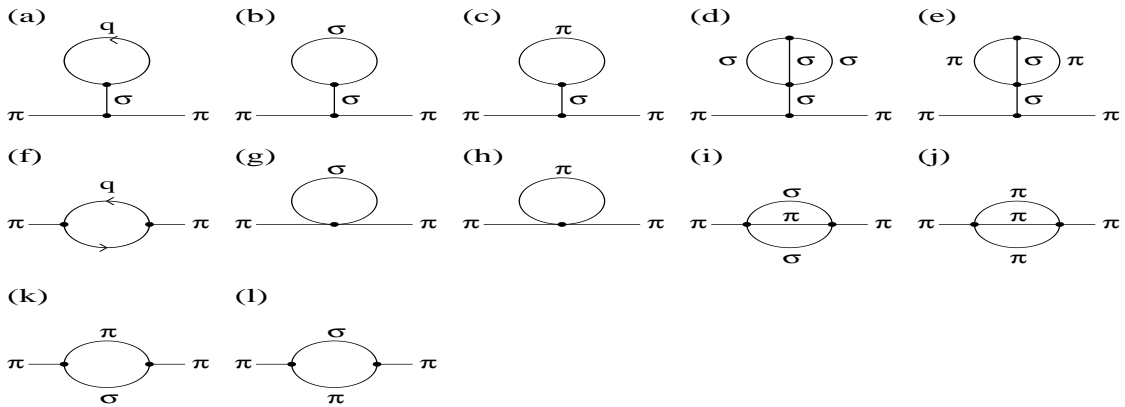


FIGURE 4. Pion mass: contributions to the π self-energy

Option 2: *dynamical generation following the strategy of Delbourgo and Scadron (DS)*

In order to illustrate the approach of DS [6, 7] we reformulate Eq. (13) slightly:

$$\begin{aligned}
S_{(4)}[\vec{\pi}^2] &= \frac{i}{2} \int d^4x \, \vec{\pi}(x)^2 \\
&\times \left[\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} \underbrace{\left\{ -\frac{8g_{\sigma\pi\pi} g N_F m_q}{m_\sigma^2} + 4g^2 N_F \right\}}_{\text{quark contributions} = 0} \right. \\
&\quad \left. + \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - \bar{m}^2} \underbrace{\left(\frac{2g_{\sigma\pi\pi}^2}{m_\sigma^2} - \lambda \right) (3 + N_\pi)}_{\text{mesonic contributions} = 0} \left(1 - \frac{\lambda}{4\pi^2} \right) \right] + \dots \quad (20)
\end{aligned}$$

As indicated above DS set individually the quark and meson contributions to the pion self-energy to zero. The sunset contributions to the pion self-energy being of two-loop order are disregarded implying the replacement $(1 - \frac{\lambda}{4\pi^2}) \rightarrow 1$ in the expression above. As a result Delbourgo and Scadron obtain the following identities:

$$0 = -\frac{8g_{\sigma\pi\pi} g N_F m_q}{m_\sigma^2} + 4g^2 N_F \quad \Rightarrow \quad g_{\sigma\pi\pi} = \frac{m_\sigma^2}{2} \frac{g}{m_q} \quad (21)$$

$$0 = \frac{2g_{\sigma\pi\pi}^2}{m_\sigma^2} - \lambda \quad \Rightarrow \quad \lambda = \frac{2}{m_\sigma^2} g_{\sigma\pi\pi}^2 = \frac{2}{m_\sigma^2} \left(\frac{m_\sigma^2}{2} \frac{g}{m_q} \right)^2 = 2g^2 \left(\frac{m_\sigma}{2m_q} \right)^2 \quad (22)$$

The two-loop relation Eq. (17), i.e. $g_{\sigma\pi\pi} = \frac{m_q}{g} \lambda$, has been derived already at one-loop order by DS via determination of the following quark-loop contributions to the couplings $g_{\sigma\pi\pi}$ and λ performing quark-loop shrinking as demonstrated in Eqs. (48) and (49):

$$g_{\sigma\pi\pi}(\text{quark-loop}) = -4g^3 N_F m_q i I_2(m_q^2) \rightarrow -4g^3 N_F m_q i \frac{i}{16\pi^2} = \frac{g^3 N_F m_q}{4\pi^2}, \quad (23)$$

$$\lambda(\text{quark-loop}) = -4g^4 N_F i I_2(m_q^2) \rightarrow -4g^4 N_F i \frac{i}{16\pi^2} = \frac{g^4 N_F}{4\pi^2}. \quad (24)$$

The right-hand side of these equations has been obtained here by renormalizing logarithmic divergencies with the help of the log.-divergent gap equation Eq. (1) choosing the renormalization scale $\bar{m} \approx m_q$. The quark-loop contributions to m_σ^2 and m_π^2 obtained by DS are given according to Eqs. (46) and (47) by

$$m_\sigma^2(\text{quark-loop}) = -8g^2 N_F m_q^2 i I_2(m_q^2) \rightarrow -8g^2 N_F m_q^2 i \frac{i}{16\pi^2} = \frac{g^2 N_F}{8\pi^2} (2m_q)^2, \quad (25)$$

$$m_\pi^2(\text{quark-loop}) = 0. \quad (26)$$

While focusing on quark-loops the dynamical generation of the $SU(2) \times SU(2)$ QLL σ M performed by DS happens to display results essentially in the Chiral Limit (CL) [22]. Hence it appears according to DS to be quite appealing and natural that the chiral limiting relation $g_{\sigma\pi\pi} \approx -e^{i(\alpha-\beta)} m_\sigma^2 / (2|f_\pi|)$ of the L σ M is resulting directly from assuming the GT relation $m_q \approx -|g|f_\pi$ on the quark-level. Moreover is the NJL-relation between m_σ and m_q in the CL $|m_\sigma| \approx |2m_q|$ [23] obviously achieved by choosing $|g^2 N_F / (8\pi^2)| \approx 1$ in Eq. (25), or equivalently $|g| = 2\pi / \sqrt{N_c}$ with $N_F = 2N_c = 6$, yielding in combination with the GT relation $m_q \approx -|g|f_\pi$ and the log.-divergent gap equation Eq. (1) at a renormalization scale $\bar{m} \approx m_q$ the important correspondence [6, 7]

$$-f_\pi \leftrightarrow -2N_F |g| i \int \frac{d^4 p}{(2\pi)^4} \frac{m_q}{(p^2 - m_q^2)^2}. \quad (27)$$

The chiral limiting NJL-relation $|m_\sigma| \approx |2m_q|$ implies then in the approach of DS due to Eq. (22) the relation $\lambda \approx 2|g|^2 e^{2i(\alpha-\beta)}$ between the quartic coupling λ and the quark-meson coupling g .

Several comments are here in order: The very interesting approach of DS has been performed unfortunately just in the limit $e^{i\alpha} = 1$ and $e^{i(\alpha-\beta)} = -1$ yielding $e^{-i\beta} = -1$. In this limit the quark-level GT relation reduces to $m_q \approx |g||f_\pi|$ implying the unphysical limit of *real-valued* quark masses. Moreover did DS derive the quark-loop contributions to m_q^2 , m_π^2 and m_σ^2 not by cancelling systematically quadratic divergencies and renormalizing afterwards the remaining log.-divergent parts of the effective action by making use of the log.-divergent gap equation Eq. (1). Instead they converted quadratic divergences into logarithmic divergences in making heavy use of Dimensional Regularization (DR) yielding on one hand as a lemma [31]

$$\int \frac{d^4 p}{(2\pi)^4} \left[\frac{m^2}{(p^2 - m^2)^2} - \frac{1}{p^2 - m^2} \right] \xrightarrow{DR} -\frac{i}{16\pi^2} m^2, \quad (28)$$

on the other hand the vanishing of massless tadpoles, i.e. $\int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2} \xrightarrow{DR} 0$, and the disappearance of quadratic divergences in unrenormalized sunset/sunrise integrals (See e.g. also the DR calculations performed in Refs. [24, 25, 26], or on p. 114 ff in Ref. [27]). From the considerations performed in Option 1 it gets very clear why dimensional regularization [28, 29, 30] and present implicit regularization [10, 11] constructed to reproduce results from dimensional regularization are not useful to perform a complete dynamical generation of QLL σ M-like quantum theories. By erasing massless tadpoles and removing quadratic divergencies in sunset/sunrise diagrams DR does not allow to obtain universal relations between coupling constants like Eqs. (14) and (15) which permit to cancel quadratic divergencies completely and simultaneously in all parts of the effective action. We would like to add here that similar problems arise also in different regularization schemes like Schwinger's regularization [4], if quadratic divergencies are not treated with due care. In the case of DS the seeming gain in disregarding — due to the use of DR — the massless pion-loop tadpole in the tadpole sum contributing to the vacuum expectation value of the scalar field spoils e.g. the cancellation of quadratic divergencies in the self-energy of the scalar field. Interestingly this problem does not

get manifestly visible in the approach of DS as they — as pointed out above — convert quadratic into logarithmic divergences with the help of DR. It has to be stressed at this place that the drawbacks of the DR approach of DS *do not merit at all* the great and fascinating implications and insights emerging from the extremely benchmarking work of DS when appreciated correctly.

Option 3: *dynamical generation of a new standard model of particle physics*

The most promising way to continue the considerations performed in this manuscript is to apply the strategy outlined in Option 1 to the quantum theory consisting of a sum on one hand of the $U(6) \times U(6)$ QLL σ M (including also vector and axial vector meson fields) to describe the asymptotically free theory of strong interactions and on the other hand the familiar theory of electroweak interactions. In such an approach the universal relations between coupling constants like Eqs. (14) and (15) to be used to cancel quadratic divergences would extend of course to all parts of the effective action stemming from the strong *and* electroweak sector of the whole theory. The quarks and anti-quarks in such an approach should be of course considered as constituent quarks rather than current quarks. A small source of mixing between scalar and pseudoscalar fields in such a theory should take care of experimental facts related e.g. to chiral symmetry breaking [21] and flavour changing neutral currents like scalar meson dominance [18, 33] in semileptonic and non-leptonic meson decays.

As we have noted already above the cancellation of quadratic divergences in such a theory is interconnecting different loop orders of the theory. This feature typically known from non-Abelian gauge theories is recovered in this manuscript already for a theory containing merely spin 1/2 Fermions and scalar and pseudoscalar spin 0 Bosons. Moreover would we like to point out that a complete knowledge of the one-point function of the Higgs field to be obtained in the same way as described here for the one-point function of the σ -meson in the $SU(2) \times SU(2)$ QLL σ M without vector and axial vector meson fields would allow to make a theoretical prediction for the vacuum expectation value of the Higgs on the basis of the knowledge of the finite part of the sunset/sunrise integral with zero external four-momentum [32].³

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³ Working out this manuscript we noted a sign mistake in the quadratically divergent term of the sunset integral in the article of Inami *et al.* [5] which made us to invoke the wrong sign in our considerations on the sunset integral in footnote 6 of Ref. [1]. Applying the corrected expressions of Appendix B to our discussion of the 1-point function of the ϕ^4 -theory in Ref. [1] we obtain a corrected Eq. (18) of Ref. [1]:

$$S_{(1)}[\phi] = \int d^4z \left(-\frac{1}{3!} g_{(1)} \right) 3i \phi_{(1)}(z) \left\{ \left(1 - \frac{2}{3} \frac{1}{16\pi^2} \lambda_{(1)} \right) I_1(m_{(1)}^2) - i \left(\frac{1}{16\pi^2} \right)^2 m_{(1)}^2 \frac{(8-C)}{3} \lambda_{(1)} \right\} + \dots \quad (18)$$

Hence the non-trivial cancellation of quadratic divergencies yields $\lambda_{(1)} = +(3/2) 16\pi^2 = +24\pi^2$ implying due to $\lambda_{(1)} = -g_{(1)}^4 / (32\pi^2 m_{(1)}^4)$ now $g_{(1)} = \pm 4\pi (+i)^{1/2} 3^{1/4} m_{(1)}$ and $g_{(1)} = \pm 4\pi (-i)^{1/2} 3^{1/4} m_{(1)}$.

A. LIST OF IMPORTANT INTEGRALS

We want to list here some important integral identities to be used in the manuscript:⁴

$$I_n(m^2) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2)^n} \stackrel{n \geq 3}{=} (-1)^n \frac{i}{16\pi^2} \frac{1}{(n-1)! m^{2n-4}}, \quad (29)$$

$$I_1(m^2) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2}, \quad (30)$$

$$I_2(m^2) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2)^2}, \quad (31)$$

$$I_3(m^2) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2)^3} = -\frac{i}{32\pi^2} \frac{1}{m^2}, \quad (32)$$

$$I_4(m^2) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2)^4} = \frac{1}{3} \frac{dI_3(m^2)}{dm^2} = +\frac{i}{96\pi^2} \frac{1}{m^4}, \quad (33)$$

$$I_{1,1}(m_1^2, m_2^2) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_1^2)(p^2 - m_2^2)}, \quad (34)$$

$$\begin{aligned} I_{2,1}(m_1^2, m_2^2) &\equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_1^2)^2(p^2 - m_2^2)} \\ &= \frac{i}{16\pi^2} \frac{1}{(m_2^2 - m_1^2)} \left(1 - \frac{m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{m_1^2} \right), \end{aligned} \quad (35)$$

$$\begin{aligned} I_{1,1,1}(m_1^2, m_2^2, m_3^2) &\equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_1^2)(p^2 - m_2^2)(p^2 - m_3^2)} \\ &= \frac{i}{16\pi^2} \frac{m_1^2 m_2^2 \ln \frac{m_1^2}{m_2^2} + m_2^2 m_3^2 \ln \frac{m_2^2}{m_3^2} + m_3^2 m_1^2 \ln \frac{m_3^2}{m_1^2}}{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}, \end{aligned} \quad (36)$$

$$\begin{aligned} I_{2,2}(m_1^2, m_2^2) &\equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_1^2)^2(p^2 - m_2^2)^2} \\ &= \frac{d}{dm_2^2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_1^2)^2(p^2 - m_2^2)} \\ &= \frac{d}{dm_2^2} \left(\frac{i}{16\pi^2} \frac{1}{(m_2^2 - m_1^2)} \left(1 - \frac{m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{m_1^2} \right) \right) \\ &= \frac{i}{16\pi^2} \frac{1}{(m_2^2 - m_1^2)^2} \left(-2 + \frac{m_1^2 + m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{m_1^2} \right). \end{aligned} \quad (37)$$

⁴ In all integrals we assume the imaginary part of the squared masses to be negative.

B. THE SUNSET / SUNRISE DIAGRAM

The leading divergent part of the sunset diagram for zero external four-momentum and equal masses has been determined in cutoff regularization by Ji-Feng Yang, Jie Zhou and Chen Wu to be [32]:⁵

$$\begin{aligned} I_{\text{sunset}}^\Lambda(m^2) &= \int^\Lambda \frac{d^4 p_1}{(2\pi)^4} \int^\Lambda \frac{d^4 p_2}{(2\pi)^4} \int^\Lambda \frac{d^4 p_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 + p_3)}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} = \\ &= \left(\frac{1}{16\pi^2} \right)^2 \left(2\Lambda^2 - \frac{3}{2} m^2 \ln^2 \left(\frac{\Lambda^2}{m^2} \right) - 3m^2 \ln \left(\frac{\Lambda^2}{m^2} \right) + C m^2 \right) + O(\Lambda^{-2}), \end{aligned} \quad (38)$$

while the integration constant C was numerically estimated by Ji-Feng Yang *et al.* to be approximately $C \simeq 4$. A numerical analysis by G. Rupp (private communication, 22.05.2006) yields $C \in [4.160805, 4.160810]$. The above result can be slightly rewritten:

$$\begin{aligned} I_{\text{sunset}}^\Lambda(m^2) &= \int^\Lambda \frac{d^4 p_1}{(2\pi)^4} \int^\Lambda \frac{d^4 p_2}{(2\pi)^4} \int^\Lambda \frac{d^4 p_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 + p_3)}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} = \\ &= \left(\frac{1}{16\pi^2} \right)^2 \left(2 \left(\Lambda^2 - m^2 \ln \left(\frac{\Lambda^2}{m^2} \right) \right) - \frac{3}{2} m^2 \left(\ln \left(\frac{\Lambda^2}{m^2} \right) - 1 \right)^2 \right. \\ &\quad \left. - 4m^2 \left(\ln \left(\frac{\Lambda^2}{m^2} \right) - 1 \right) + m^2 \left(C - \frac{5}{2} \right) \right) + O(\Lambda^{-2}) \\ &\rightarrow 2 \frac{i}{16\pi^2} I_1(m^2) + \frac{3}{2} m^2 I_2(m^2)^2 + 4 m^2 \frac{i}{16\pi^2} I_2(m^2) - \left(\frac{i}{16\pi^2} \right)^2 m^2 \left(C - \frac{5}{2} \right). \end{aligned} \quad (39)$$

The last line displays manifestly the most divergent part of the massive sunset diagram at zero external four-momentum in an regularization scheme independent manner.

Now we may apply the renormalization procedure of Delbourgo and Scadron replacing the log.-divergent Bosonic one-loop integral at some renormalization scale \bar{m} ($\simeq \hat{m} = m_q$) by the finite number $+\frac{i}{16\pi^2}$ (by adding a suitable counterterm), i.e. $I_2(\bar{m}^2) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - \bar{m}^2)^2} \rightarrow +\frac{i}{16\pi^2}$ being known as *log.-divergent gap equation*. Then the sunset integral in regularization scheme indepent representation reduces to:

$$\begin{aligned} I_{\text{sunset}}(\bar{m}^2) &= \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 + p_3)}{(p_1^2 - \bar{m}^2)(p_2^2 - \bar{m}^2)(p_3^2 - \bar{m}^2)} \\ &\rightarrow 2 \frac{i}{16\pi^2} I_1(\bar{m}^2) - \left(\frac{1}{16\pi^2} \right)^2 \bar{m}^2 (8 - C). \end{aligned} \quad (40)$$

⁵ For a discussion of the finite part of the sunset/sunrise integral for non-zero external four-momentum on the basis of implicit renormalization see e.g. Ref. [11].

The situation is more involved if the equal mass sunset integral isn't evaluated at the renormalization scale \bar{m} , yet at some arbitrary mass m . Using Eqs. (4) and (5) we obtain:

$$\begin{aligned}
I_{\text{sunset}}(m^2) &= \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 + p_3)}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \\
&\rightarrow 2 \frac{i}{16\pi^2} I_1(m^2) + \frac{3}{2} m^2 I_2(m^2)^2 + 4 m^2 \frac{i}{16\pi^2} I_2(m^2) - \left(\frac{i}{16\pi^2} \right)^2 m^2 \left(C - \frac{5}{2} \right) \\
&\rightarrow 2 \frac{i}{16\pi^2} \left(I_1(\bar{m}^2) + \frac{i}{16\pi^2} (m^2 - \bar{m}^2) \left(2 - \frac{m^2}{m^2 - \bar{m}^2} \ln \frac{m^2}{\bar{m}^2} \right) \right) \\
&\quad + \frac{3}{2} m^2 \left(\frac{i}{16\pi^2} \right)^2 \left(1 - \ln \frac{m^2}{\bar{m}^2} \right)^2 + 4 m^2 \left(\frac{i}{16\pi^2} \right)^2 \left(1 - \ln \frac{m^2}{\bar{m}^2} \right) \\
&\quad + \left(\frac{i}{16\pi^2} \right)^2 m^2 \left(\frac{5}{2} - C \right) \\
&= \frac{i}{16\pi^2} \left\{ 2 I_1(\bar{m}^2) + \frac{i}{16\pi^2} \left(\frac{3}{2} m^2 \left(\ln \frac{m^2}{\bar{m}^2} \right)^2 - 9 m^2 \ln \frac{m^2}{\bar{m}^2} + m^2 (12 - C) - 4 \bar{m}^2 \right) \right\}.
\end{aligned} \tag{41}$$

In order to complete our discussion it would be necessary to consider some special cases of the more general class of sunset-like integrals with zero external four-momentum of the following type:

$$\begin{aligned}
I_{n_1, n_2, n_3}^{\text{sunset}}(m_1^2, m_2^2, m_3^2) &\equiv \\
&\equiv \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 + p_3)}{(p_1^2 - m_1^2)^{n_1} (p_2^2 - m_2^2)^{n_2} (p_3^2 - m_3^2)^{n_3}}
\end{aligned} \tag{42}$$

yielding of course as a special case also the above discussed integral $I_{\text{sunset}}(m^2) = I_{1,1,1}^{\text{sunset}}(m^2, m^2, m^2)$. Unfortunately there is lacking yet an analysis of the the quadratically divergent integral $I_{1,1,1}^{\text{sunset}}(m'^2, m^2, m^2)$ beyond its quadratic divergence in the same spirit as it has been provided above for the integral $I_{\text{sunset}}(m^2)$:

$$\begin{aligned}
I_{1,1,1}^{\text{sunset}}(m'^2, m^2, m^2) &= \\
&= \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 + p_3)}{(p_1^2 - m'^2)(p_2^2 - m^2)(p_3^2 - m^2)} \\
&= I_{\text{sunset}}(m^2) + (m'^2 - m^2) \times \\
&\quad \times \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 + p_3)}{(p_1^2 - m'^2)(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)}.
\end{aligned} \tag{43}$$

As it can be seen by inspection of Appendix C the integral $I_{1,1,1}^{\text{sunset}}(m'^2, m^2, m^2)$ appears at various places in the effective actions under consideration.

C. EFFECTIVE ACTIONS

The effective action $S_{(1)}[\sigma]$ of the scalar one-point function consisting of the contributions illustrated in Fig. 1 is given by [1]:

$$\begin{aligned}
S_{(1)}[\sigma] &= S_{(1a)}[\sigma] + S_{(1b)}[\sigma] + S_{(1c)}[\sigma] + S_{(1d)}[\sigma] + S_{(1e)}[\sigma] \\
&= \int d^4x \, \sigma(x) \left\{ \langle 0 | T \left[g \overline{q_+^c}(x) q_-(x) + 3 g_{\sigma\pi\pi} \sigma(x)^2 + N_\pi g_{\sigma\pi\pi} \pi(x)^2 \right] | 0 \rangle \right. \\
&\quad \left. + i \int d^4x' \left(-\frac{\lambda}{4} \right) g_{\sigma\pi\pi} \langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle \right. \\
&\quad \left. \times 8 \left(3 \langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle^2 + N_\pi \langle 0 | T[\pi(x) \pi(x')] | 0 \rangle^2 \right) \right\} \\
&= \int d^4x \, \sigma(x) \left\{ i \int \frac{d^4p}{(2\pi)^4} \left(-\frac{4 g N_F m_q}{p^2 - m_q^2} + \frac{3 g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
&\quad \left. + i \int d^4x' \, i^3 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} e^{-i(p_1+p_2+p_3) \cdot (x-x')} \right. \\
&\quad \left. \times \left(-\frac{\lambda}{4} \right) \frac{8 g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \right\} \\
&= \int d^4x \, \sigma(x) \, i \left\{ \int \frac{d^4p}{(2\pi)^4} \left(-\frac{4 g N_F m_q}{p^2 - m_q^2} + \frac{3 g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
&\quad \left. + i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \right. \\
&\quad \left. \times \frac{2 \lambda g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \right\} \\
&\quad + \text{non-local terms} \\
&= \int d^4x \, \sigma(x) \, i \left\{ -4 g N_F m_q I_1(m_q^2) + 3 g_{\sigma\pi\pi} I_1(m_\sigma^2) + N_\pi g_{\sigma\pi\pi} I_1(m_\pi^2) \right. \\
&\quad \left. + 2 \lambda g_{\sigma\pi\pi} i \left(3 I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\sigma^2, m_\sigma^2) + N_\pi I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\pi^2, m_\pi^2) \right) \right\} \\
&\quad + \text{non-local terms} . \tag{44}
\end{aligned}$$

The effective action $S_{(2)}[\bar{q}q]$ for the two-point function of the (anti)quarks consisting of the contributions illustrated in Fig. 2 is given by [1]:

$$\begin{aligned}
S_{(2)}[\bar{q}q] &= \\
&= S_{(2a)}[\bar{q}q] + S_{(2b)}[\bar{q}q] + S_{(2c)}[\bar{q}q] + S_{(2d)}[\bar{q}q] + S_{(2e)}[\bar{q}q] + S_{(2f)}[\bar{q}q] + S_{(2g)}[\bar{q}q] \\
&= \frac{i}{2} \int d^4x \, \overline{q_+^c}(x) q_-(x) \int d^4z \, \langle 0 | T[\sigma(x) \sigma(z)] | 0 \rangle
\end{aligned}$$

$$\begin{aligned}
& \times 2g \left\{ \langle 0 | T \left[g \bar{q}_+^c(z) q_-(z) + 3g_{\sigma\pi\pi} \sigma(z)^2 + N_\pi g_{\sigma\pi\pi} \pi(z)^2 \right] | 0 \rangle \right. \\
& \quad + i \int d^4 z' \left(-\frac{\lambda}{4} \right) g_{\sigma\pi\pi} \langle 0 | T[\sigma(z) \sigma(z')] | 0 \rangle \\
& \quad \left. \times 8 \left(3 \langle 0 | T[\sigma(z) \sigma(z')] | 0 \rangle^2 + N_\pi \langle 0 | T[\pi(z) \pi(z')] | 0 \rangle^2 \right) \right\} \\
& + \frac{i}{2} \int d^4 x \int d^4 x' 2g^2 \left\{ \bar{q}_+^c(x) \langle 0 | T \left[q_-(x) \bar{q}_+^c(x') \right] | 0 \rangle q_-(x') \langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle \right. \\
& \quad \left. + N_\pi \bar{q}_+^c(x) i \gamma_5 \langle 0 | T \left[q_-(x) \bar{q}_+^c(x') \right] | 0 \rangle i \gamma_5 q_-(x') \langle 0 | T[\pi(x) \pi(x')] | 0 \rangle \right\} \\
& = \frac{i}{2} \int d^4 x \bar{q}_+^c(x) q_-(x) \int d^4 z i \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot (x-z)} \frac{1}{q^2 - m_\sigma^2} \\
& \quad \times 2g \left\{ i \int \frac{d^4 p}{(2\pi)^4} \left(-\frac{4g N_F m_q}{p^2 - m_q^2} + \frac{3g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
& \quad + i \int d^4 z' i^3 \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} e^{-i(p_1+p_2+p_3) \cdot (z-z')} \\
& \quad \times \left(-\frac{\lambda}{4} \right) \frac{8g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \Big\} \\
& + \frac{i}{2} \int d^4 x \int d^4 x' i^2 \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} e^{-i(p_1+p_2) \cdot (x-x')} \\
& \quad \times 2g^2 \bar{q}_+^c(x) \left(\frac{(\not{p}_1 + m_q)}{(p_1^2 - m_q^2)(p_2^2 - m_\sigma^2)} + \frac{N_\pi (\not{p}_1 - m_q)}{(p_1^2 - m_q^2)(p_2^2 - m_\pi^2)} \right) q_-(x') \\
& = \frac{i}{2} \int d^4 x \bar{q}_+^c(x) q_-(x) \frac{2g}{m_\sigma^2} \left\{ \int \frac{d^4 p}{(2\pi)^4} \left(-\frac{4g N_F m_q}{p^2 - m_q^2} + \frac{3g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
& \quad + i \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \\
& \quad \times \frac{2\lambda g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \Big\} \\
& - \frac{i}{2} \int d^4 x \bar{q}_+^c(x) q_-(x) 2g^2 m_q \int \frac{d^4 p}{(2\pi)^4} \left(\frac{1}{(p^2 - m_q^2)(p^2 - m_\sigma^2)} - \frac{N_\pi}{(p^2 - m_q^2)(p^2 - m_\pi^2)} \right) \\
& + \text{non-local terms} \\
& = \frac{i}{2} \int d^4 x \bar{q}_+^c(x) q_-(x) \frac{2g}{m_\sigma^2} \left\{ -4g N_F m_q I_1(m_q^2) + 3g_{\sigma\pi\pi} I_1(m_\sigma^2) + N_\pi g_{\sigma\pi\pi} I_1(m_\pi^2) \right. \\
& \quad \left. + 2\lambda g_{\sigma\pi\pi} i \left(3 I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\sigma^2, m_\sigma^2) + N_\pi I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\pi^2, m_\pi^2) \right) \right\} \\
& - \frac{i}{2} \int d^4 x \bar{q}_+^c(x) q_-(x) 2g^2 m_q \left(I_{1,1}(m_q^2, m_\sigma^2) - N_\pi I_{1,1}(m_q^2, m_\pi^2) \right) \\
& + \text{non-local terms} .
\end{aligned} \tag{45}$$

The effective action $S_{(3)}[\sigma^2]$ for the two-point function of the σ consisting of the contributions illustrated in Fig. 3 is given by [1]:

$$\begin{aligned}
S_{(3)}[\sigma^2] = & \\
& + S_{(3a)}[\sigma^2] + S_{(3b)}[\sigma^2] + S_{(3c)}[\sigma^2] + S_{(3d)}[\sigma^2] + S_{(3e)}[\sigma^2] \\
& + S_{(3f)}[\sigma^2] + S_{(3g)}[\sigma^2] + S_{(3h)}[\sigma^2] + S_{(3i)}[\sigma^2] + S_{(3j)}[\sigma^2] + S_{(3k)}[\sigma^2] + S_{(3l)}[\sigma^2] \\
= & \frac{i}{2} \int d^4x \sigma(x)^2 \int d^4z \langle 0 | T[\sigma(x) \sigma(z)] | 0 \rangle \\
& \times 6 g_{\sigma\pi\pi} \left\{ \langle 0 | T \left[g \bar{q}_+^c(z) q_-(z) + 3 g_{\sigma\pi\pi} \sigma(z)^2 + N_\pi g_{\sigma\pi\pi} \pi(z)^2 \right] | 0 \rangle \right. \\
& \quad + i \int d^4z' \left(-\frac{\lambda}{4} \right) g_{\sigma\pi\pi} \langle 0 | T[\sigma(z) \sigma(z')] | 0 \rangle \\
& \quad \left. \times 8 \left(3 \langle 0 | T[\sigma(z) \sigma(z')] | 0 \rangle^2 + N_\pi \langle 0 | T[\pi(z) \pi(z')] | 0 \rangle^2 \right) \right\} \\
& + \frac{i}{2} \int d^4x \int d^4x' \sigma(x) \sigma(x') g^2 \langle 0 | T \left[\bar{q}_+^c(x) q_-(x) \bar{q}_+^c(x') q_-(x') \right] | 0 \rangle_c \\
& + \int d^4x \sigma(x)^2 2 \left(-\frac{\lambda}{4} \right) \left(3 \langle 0 | T[\sigma(x) \sigma(x)] | 0 \rangle + N_\pi \langle 0 | T[\pi(x) \pi(x)] | 0 \rangle \right) \\
& + \frac{i}{2} \int d^4x \int d^4x' \sigma(x) \sigma(x') \langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle \\
& \quad \times 2 \lambda^2 \left(3 \langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle^2 + N_\pi \langle 0 | T[\pi(x) \pi(x')] | 0 \rangle^2 \right) \\
& + \frac{i}{2} \int d^4x \int d^4x' \sigma(x) \sigma(x') \\
& \quad \times 2 g_{\sigma\pi\pi}^2 \left(9 \langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle^2 + N_\pi \langle 0 | T[\pi(x) \pi(x')] | 0 \rangle^2 \right) \\
= & \frac{i}{2} \int d^4x \sigma(x)^2 \int d^4z i \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (x-z)} \frac{1}{q^2 - m_\sigma^2} \\
& \times 6 g_{\sigma\pi\pi} \left\{ i \int \frac{d^4p}{(2\pi)^4} \left(-\frac{4 g N_F m_q}{p^2 - m_q^2} + \frac{3 g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
& \quad + i \int d^4z' i^3 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} e^{-i(p_1+p_2+p_3) \cdot (z-z')} \\
& \quad \left. \times \left(-\frac{\lambda}{4} \right) \frac{8 g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \right\} \\
& - \frac{i}{2} \int d^4x \int d^4x' \sigma(x) \sigma(x') i^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{-i(p_1+p_2) \cdot (x-x')} \frac{4 g^2 N_F (p_1 \cdot p_2 + m_q^2)}{(p_1^2 - m_q^2)(p_2^2 - m_q^2)} \\
& + \int d^4x \sigma(x)^2 2 \left(-\frac{\lambda}{4} \right) i \int \frac{d^4p}{(2\pi)^4} \left(\frac{3}{p^2 - m_\sigma^2} + \frac{N_\pi}{p^2 - m_\pi^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{i}{2} \int d^4x \int d^4x' \sigma(x) \sigma(x') i^3 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} e^{-i(p_1+p_2+p_3) \cdot (x-x')} \\
& \quad \times \frac{2\lambda^2}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \\
& + \frac{i}{2} \int d^4x \int d^4x' \sigma(x) \sigma(x') i^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{-i(p_1+p_2) \cdot (x-x')} \\
& \quad \times 2g_{\sigma\pi\pi}^2 \left(\frac{9}{(p_1^2 - m_\sigma^2)(p_2^2 - m_\sigma^2)} + \frac{N_\pi}{(p_1^2 - m_\pi^2)(p_2^2 - m_\pi^2)} \right) \\
& = \frac{i}{2} \int d^4x \sigma(x)^2 \frac{6g_{\sigma\pi\pi}}{m_\sigma^2} \left\{ \int \frac{d^4p}{(2\pi)^4} \left(-\frac{4g N_F m_q}{p^2 - m_q^2} + \frac{3g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
& \quad + i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \\
& \quad \times \frac{2\lambda g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \Big\} \\
& + \frac{i}{2} \int d^4x \sigma(x)^2 4g^2 N_F \int \frac{d^4p}{(2\pi)^4} \frac{p^2 + m_q^2}{(p^2 - m_q^2)^2} \\
& - \frac{i}{2} \int d^4x \sigma(x)^2 \lambda \int \frac{d^4p}{(2\pi)^4} \left(\frac{3}{p^2 - m_\sigma^2} + \frac{N_\pi}{p^2 - m_\pi^2} \right) \\
& - \frac{i}{2} \int d^4x \sigma(x)^2 i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \\
& \quad \times \frac{2\lambda^2}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \\
& - \frac{i}{2} \int d^4x \sigma(x)^2 2g_{\sigma\pi\pi}^2 \int \frac{d^4p}{(2\pi)^4} \left(\frac{9}{(p^2 - m_\sigma^2)^2} + \frac{N_\pi}{(p^2 - m_\pi^2)^2} \right) + \text{non-local terms} \\
& = \frac{i}{2} \int d^4x \sigma(x)^2 \frac{6g_{\sigma\pi\pi}}{m_\sigma^2} \left\{ -4g N_F m_q I_1(m_q^2) + 3g_{\sigma\pi\pi} I_1(m_\sigma^2) + N_\pi g_{\sigma\pi\pi} I_1(m_\pi^2) \right. \\
& \quad \left. + 2\lambda g_{\sigma\pi\pi} i \left(3 I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\sigma^2, m_\sigma^2) + N_\pi I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\pi^2, m_\pi^2) \right) \right\} \\
& + \frac{i}{2} \int d^4x \sigma(x)^2 4g^2 N_F \left(I_1(m_q^2) + 2m_q^2 I_2(m_q^2) \right) \\
& - \frac{i}{2} \int d^4x \sigma(x)^2 \lambda \left(3 I_1(m_\sigma^2) + N_\pi I_1(m_\pi^2) \right) \\
& - \frac{i}{2} \int d^4x \sigma(x)^2 2\lambda^2 i \left(3 I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\sigma^2, m_\sigma^2) + N_\pi I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\pi^2, m_\pi^2) \right) \\
& - \frac{i}{2} \int d^4x \sigma(x)^2 2g_{\sigma\pi\pi}^2 \left(9 I_2(m_\sigma^2) + N_\pi I_2(m_\pi^2) \right) + \text{non-local terms} . \tag{46}
\end{aligned}$$

Analogously the effective action $S_{(4)}[\vec{\pi}^2]$ for the two-point function of the π consisting of the contributions illustrated in Fig. 4 is given by [1]:

$$\begin{aligned}
S_{(4)}[\vec{\pi}^2] = & \\
& + S_{(4a)}[\vec{\pi}^2] + S_{(4b)}[\vec{\pi}^2] + S_{(4c)}[\vec{\pi}^2] + S_{(4d)}[\vec{\pi}^2] + S_{(4e)}[\vec{\pi}^2] \\
& + S_{(4f)}[\vec{\pi}^2] + S_{(4g)}[\vec{\pi}^2] + S_{(4h)}[\vec{\pi}^2] + S_{(4i)}[\vec{\pi}^2] + S_{(4j)}[\vec{\pi}^2] + S_{(4k)}[\vec{\pi}^2] + S_{(4l)}[\vec{\pi}^2] \\
= & \frac{i}{2} \int d^4x \vec{\pi}(x)^2 \int d^4z \langle 0 | T[\sigma(x) \sigma(z)] | 0 \rangle \\
& \times 2g_{\sigma\pi\pi} \left\{ \langle 0 | T \left[g \overline{q}_+^c(z) q_-(z) + 3g_{\sigma\pi\pi} \sigma(z)^2 + N_\pi g_{\sigma\pi\pi} \pi(z)^2 \right] | 0 \rangle \right. \\
& \quad + i \int d^4z' \left(-\frac{\lambda}{4} \right) g_{\sigma\pi\pi} \langle 0 | T[\sigma(z) \sigma(z')] | 0 \rangle \\
& \quad \left. \times 8 \left(3 \langle 0 | T[\sigma(z) \sigma(z')] | 0 \rangle^2 + N_\pi \langle 0 | T[\pi(z) \pi(z')] | 0 \rangle^2 \right) \right\} \\
& + \frac{i}{2} \int d^4x \int d^4x' \vec{\pi}(x) \cdot \vec{\pi}(x') g^2 \langle 0 | T \left[\overline{q}_+^c(x) i \gamma_5 q_-(x) \overline{q}_+^c(x') i \gamma_5 q_-(x') \right] | 0 \rangle_c \\
& + \int d^4x \vec{\pi}(x)^2 2 \left(-\frac{\lambda}{4} \right) \left(\langle 0 | T[\sigma(x) \sigma(x)] | 0 \rangle + (N_\pi + 2) \langle 0 | T[\pi(x) \pi(x)] | 0 \rangle \right) \\
& + \frac{i}{2} \int d^4x \int d^4x' \vec{\pi}(x) \cdot \vec{\pi}(x') \langle 0 | T[\pi(x) \pi(x')] | 0 \rangle \\
& \quad \times 2\lambda^2 \left(\langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle^2 + (N_\pi + 2) \langle 0 | T[\pi(x) \pi(x')] | 0 \rangle^2 \right) \\
& + \frac{i}{2} \int d^4x \int d^4x' \vec{\pi}(x) \cdot \vec{\pi}(x') 4g_{\sigma\pi\pi}^2 \langle 0 | T[\sigma(x) \sigma(x')] | 0 \rangle \langle 0 | T[\pi(x) \pi(x')] | 0 \rangle \\
= & \frac{i}{2} \int d^4x \vec{\pi}(x)^2 \int d^4z i \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (x-z)} \frac{1}{q^2 - m_\sigma^2} \\
& \times 2g_{\sigma\pi\pi} \left\{ i \int \frac{d^4p}{(2\pi)^4} \left(-\frac{4g N_F m_q}{p^2 - m_q^2} + \frac{3g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
& \quad + i \int d^4z' i^3 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} e^{-i(p_1+p_2+p_3) \cdot (z-z')} \\
& \quad \left. \times \left(-\frac{\lambda}{4} \right) \frac{8g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \right\} \\
& - \frac{i}{2} \int d^4x \int d^4x' \vec{\pi}(x) \cdot \vec{\pi}(x') i^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{-i(p_1+p_2) \cdot (x-x')} \frac{4g^2 N_F (p_1 \cdot p_2 - m_q^2)}{(p_1^2 - m_q^2)(p_2^2 - m_q^2)} \\
& + \int d^4x \vec{\pi}(x)^2 2 \left(-\frac{\lambda}{4} \right) i \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{p^2 - m_\sigma^2} + \frac{N_\pi + 2}{p^2 - m_\pi^2} \right) \\
& + \frac{i}{2} \int d^4x \int d^4x' \vec{\pi}(x) \cdot \vec{\pi}(x') i^3 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} e^{-i(p_1+p_2+p_3) \cdot (x-x')}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{2\lambda^2}{p_1^2 - m_\pi^2} \left(\frac{1}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi + 2}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \\
& + \frac{i}{2} \int d^4x \int d^4x' \vec{\pi}(x) \cdot \vec{\pi}(x') i^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{-i(p_1+p_2) \cdot (x-x')} \\
& \quad \times 4g_{\sigma\pi\pi}^2 \frac{1}{(p_1^2 - m_\sigma^2)(p_2^2 - m_\pi^2)} \\
& = \frac{i}{2} \int d^4x \vec{\pi}(x)^2 \frac{2g_{\sigma\pi\pi}}{m_\sigma^2} \left\{ \int \frac{d^4p}{(2\pi)^4} \left(-\frac{4g N_F m_q}{p^2 - m_q^2} + \frac{3g_{\sigma\pi\pi}}{p^2 - m_\sigma^2} + \frac{N_\pi g_{\sigma\pi\pi}}{p^2 - m_\pi^2} \right) \right. \\
& \quad + i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \\
& \quad \times \frac{2\lambda g_{\sigma\pi\pi}}{p_1^2 - m_\sigma^2} \left(\frac{3}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \Big\} \\
& + \frac{i}{2} \int d^4x \vec{\pi}(x)^2 4g^2 N_F \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m_q^2} \\
& - \frac{i}{2} \int d^4x \vec{\pi}(x)^2 \lambda \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{p^2 - m_\sigma^2} + \frac{N_\pi + 2}{p^2 - m_\pi^2} \right) \\
& - \frac{i}{2} \int d^4x \vec{\pi}(x)^2 i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \\
& \quad \times \frac{2\lambda^2}{p_1^2 - m_\pi^2} \left(\frac{1}{(p_2^2 - m_\sigma^2)(p_3^2 - m_\sigma^2)} + \frac{N_\pi + 2}{(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)} \right) \\
& - \frac{i}{2} \int d^4x \vec{\pi}(x)^2 4g_{\sigma\pi\pi}^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m_\sigma^2)(p^2 - m_\pi^2)} + \text{non-local terms} \\
& = \frac{i}{2} \int d^4x \vec{\pi}(x)^2 \frac{2g_{\sigma\pi\pi}}{m_\sigma^2} \left\{ -4g N_F m_q I_1(m_q^2) + 3g_{\sigma\pi\pi} I_1(m_\sigma^2) + N_\pi g_{\sigma\pi\pi} I_1(m_\pi^2) \right. \\
& \quad \left. + 2\lambda g_{\sigma\pi\pi} i \left(3 I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\sigma^2, m_\sigma^2) + N_\pi I_{1,1,1}^{\text{sunset}}(m_\sigma^2, m_\pi^2, m_\pi^2) \right) \right\} \\
& + \frac{i}{2} \int d^4x \vec{\pi}(x)^2 4g^2 N_F I_1(m_q^2) \\
& - \frac{i}{2} \int d^4x \vec{\pi}(x)^2 \lambda \left(I_1(m_\sigma^2) + (N_\pi + 2) I_1(m_\pi^2) \right) \\
& - \frac{i}{2} \int d^4x \vec{\pi}(x)^2 2\lambda^2 i \left(I_{1,1,1}^{\text{sunset}}(m_\pi^2, m_\sigma^2, m_\sigma^2) + (N_\pi + 2) I_{1,1,1}^{\text{sunset}}(m_\pi^2, m_\pi^2, m_\pi^2) \right) \\
& - \frac{i}{2} \int d^4x \vec{\pi}(x)^2 4g_{\sigma\pi\pi}^2 I_{1,1}(m_\sigma^2, m_\pi^2) + \text{non-local terms} . \tag{47}
\end{aligned}$$

For convenience we want to recall here also the derivation of the quark-loop contribution to the effective actions for the $\sigma\pi\pi$ - and the π^4 -interactions [1]:

$$\begin{aligned}
S_{\text{quark-loop}}[\sigma\pi^2] &= \frac{i^2}{2!} \int d^4x \int d^4x_1 \int d^4x_2 (-2) g^3 \text{tr} \left[\sigma(x) \langle 0 | T[q_-(x) \bar{q}_+^c(x_1)] | 0 \rangle_c \right. \\
&\quad \left. \times i\gamma_5 \vec{\tau} \cdot \vec{\pi}(x_1) \langle 0 | T[q_-(x_1) \bar{q}_+^c(x_2)] | 0 \rangle_c i\gamma_5 \vec{\tau} \cdot \vec{\pi}(x_2) \langle 0 | T[q_-(x_2) \bar{q}_+^c(x)] | 0 \rangle_c \right] \\
&= \frac{i^2}{2!} \int d^4x \int d^4x_1 \int d^4x_2 (-2) g^3 N_F \sigma(x) \vec{\pi}(x_1) \cdot \vec{\pi}(x_2) \\
&\quad \times i^3 \int \frac{d^4p_{01}}{(2\pi)^4} \int \frac{d^4p_{12}}{(2\pi)^4} \int \frac{d^4p_{20}}{(2\pi)^4} e^{-ip_{01} \cdot (x-x_1)} e^{-ip_{12} \cdot (x_1-x_2)} e^{-ip_{20} \cdot (x_2-x)} \\
&\quad \times \text{tr} \left[\frac{\not{p}_{01} + m_q}{p_{01}^2 - m_q^2} i\gamma_5 \frac{\not{p}_{12} + m_q}{p_{12}^2 - m_q^2} i\gamma_5 \frac{\not{p}_{20} + m_q}{p_{20}^2 - m_q^2} \right] \\
&= i \int d^4x (-1) g^3 N_F \sigma(x) \vec{\pi}(x)^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[\frac{\not{p} + m_q}{p^2 - m_q^2} i\gamma_5 \frac{\not{p} + m_q}{p^2 - m_q^2} i\gamma_5 \frac{\not{p} + m_q}{p^2 - m_q^2} \right] \\
&\quad + \text{non-local terms} \\
&= i \int d^4x (-4) g^3 N_F m_q \sigma(x) \vec{\pi}(x)^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m_q^2)^2} + \text{non-local terms} \\
&= i \int d^4x (-4) g^3 N_F m_q \sigma(x) \vec{\pi}(x)^2 I_2(m_q^2) + \text{non-local terms} , \tag{48}
\end{aligned}$$

$$\begin{aligned}
S_{\text{quark-loop}}[(\vec{\pi}^2)^2] &= \frac{i^3}{4!} \int d^4x \int d^4x_1 \int d^4x_2 \int d^4x_3 (-6) g^4 \\
&\quad \times \text{tr} \left[i\gamma_5 \vec{\tau} \cdot \vec{\pi}(x) \langle 0 | T[q_-(x) \bar{q}_+^c(x_1)] | 0 \rangle_c i\gamma_5 \vec{\tau} \cdot \vec{\pi}(x_1) \langle 0 | T[q_-(x_1) \bar{q}_+^c(x_2)] | 0 \rangle_c \right. \\
&\quad \left. \times i\gamma_5 \vec{\tau} \cdot \vec{\pi}(x_2) \langle 0 | T[q_-(x_2) \bar{q}_+^c(x_3)] | 0 \rangle_c i\gamma_5 \vec{\tau} \cdot \vec{\pi}(x_3) \langle 0 | T[q_-(x_3) \bar{q}_+^c(x)] | 0 \rangle_c \right] \\
&= \frac{i^3}{4!} \int d^4x \int d^4x_1 \int d^4x_2 (-6) g^4 N_F \vec{\pi}(x) \cdot \vec{\pi}(x_1) \vec{\pi}(x_2) \cdot \vec{\pi}(x_3) \\
&\quad \times i^4 \int \frac{d^4p_{01}}{(2\pi)^4} \int \frac{d^4p_{12}}{(2\pi)^4} \int \frac{d^4p_{23}}{(2\pi)^4} \int \frac{d^4p_{30}}{(2\pi)^4} e^{-ip_{01} \cdot (x-x_1)} e^{-ip_{12} \cdot (x_1-x_2)} e^{-ip_{23} \cdot (x_2-x_3)} \\
&\quad \times e^{-ip_{30} \cdot (x_3-x)} \text{tr} \left[i\gamma_5 \frac{\not{p}_{01} + m_q}{p_{01}^2 - m_q^2} i\gamma_5 \frac{\not{p}_{12} + m_q}{p_{12}^2 - m_q^2} i\gamma_5 \frac{\not{p}_{23} + m_q}{p_{23}^2 - m_q^2} i\gamma_5 \frac{\not{p}_{30} + m_q}{p_{30}^2 - m_q^2} \right] \\
&= \frac{i}{4} \int d^4x g^4 N_F (\vec{\pi}(x)^2)^2 \int \frac{d^4p}{(2\pi)^4} \\
&\quad \times \text{tr} \left[i\gamma_5 \frac{\not{p} + m_q}{p^2 - m_q^2} i\gamma_5 \frac{\not{p} + m_q}{p^2 - m_q^2} i\gamma_5 \frac{\not{p} + m_q}{p^2 - m_q^2} i\gamma_5 \frac{\not{p} + m_q}{p^2 - m_q^2} \right] + \text{non-local terms}
\end{aligned}$$

$$\begin{aligned}
&= i \int d^4x g^4 N_F \left(\vec{\pi}(x)^2 \right)^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m_q^2)^2} + \text{non-local terms} \\
&= i \int d^4x g^4 N_F \left(\vec{\pi}(x)^2 \right)^2 I_2(m_q^2) + \text{non-local terms} .
\end{aligned} \tag{49}$$

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